Question 1. Suppose Y follows a logistic distribution. The density function for Y is

$$f_Y(y) = \frac{e^y}{(1+e^y)^2}, -\infty < y < \infty$$

Find the distribution of $U = \frac{1}{1+e^{-Y}}$.

Question 2. Suppose $Y_1, \ldots, Y_n \sim f_Y(y)$, where the density is

$$f_Y(y) = p_1 \frac{1}{\sqrt{2\pi\sigma_1^2}} exp\left(-\frac{(y-\mu)^2}{2\sigma_1^2}\right) + p_2 \frac{1}{\sqrt{2\pi\sigma_2^2}} exp\left(-\frac{(y-2\mu)^2}{2\sigma_2^2}\right), -\infty < y < \infty$$

where $p_1, p_2, \sigma_1^2, \sigma_2^2$ are known. Find MOM of μ.

Question 3. Suppose Y follows a shifted exponential distribution with density

$$f_Y(y) = e^{-(y-\theta)}, y > \theta$$

a. Find the MOM of θ, denoted as $\widetilde{\theta}$.

b. Find the MLE of θ, denoted as $\widehat{\theta}$. Also find the MLE of $E(Y)$.

c. Are the estimators you have obtained in (a) and (b) biased? If they are biased, then modify the estimators to obtain unbiased estimators.

d. Find the MSE of $\widetilde{\theta}$ and $\widehat{\theta}$.

e. Find a sufficient statistic for θ

f. Find the MVUE of θ

g. Suppose a random sample $Y_1, \ldots, Y_n$ follow the shifted exponential distribution, find the distribution of $Y_{(n)}$.

Question 4. Suppose that $Y_1, \ldots, Y_n$ is a sample of size n from a gamma-distributed population with $\alpha = 2$ and unknown β.

a. Use the method of moment-generating function to show that $2\sum_1^n Y_i/\beta$ is a pivotal quantity and has a χ^2 distribution with $4n$ df.

Question 1 :

Use the transformation technique.

$$U = \frac{1}{1+e^{-Y}} \Rightarrow Y = g^{-1}(u) = \log\left(\frac{u}{1-u}\right)$$

$$\Rightarrow \frac{dg^{-1}(u)}{du} = \frac{1}{u(1-u)}, \quad 0<u<1$$

$$\Rightarrow f_U(u) = \frac{e^{\log\frac{u}{1-u}}}{\left(1+e^{\log\frac{u}{1-u}}\right)^2} \cdot \frac{1}{u(1-u)}$$

$$= u(1-u) \cdot \frac{1}{u(1-u)}$$

$$= 1$$

Therefore, U has uniform distribution

$$U \sim \text{Unif}(0,1)$$

Question 2 ::

Set $EY = \bar{Y}$, where

$$EY = \int_{-\infty}^{\infty} y \cdot p_1 \cdot \underbrace{\frac{1}{\sqrt{2\pi\sigma_1^2}} \exp\left(-\frac{(y-\mu)^2}{2\sigma_1^2}\right)}_{N(\mu,\, \sigma_1^2)} + y \cdot p_2 \cdot \underbrace{\frac{1}{\sqrt{2\pi\sigma_2^2}} \exp\left(-\frac{(y-2\mu)^2}{2\sigma_2^2}\right)}_{N(2\mu,\, \sigma_2^2)} \, dy$$

$$= \mu p_1 + 2\mu p_2$$

$$= \mu(p_1 + 2p_2)$$

Therefore, $\hat{\theta}_{mom} = \dfrac{\cancel{p_1 + 2p_2}\ \bar{Y}}{p_1 + 2p_2}$

Question 3:

(a): Set $EY = \bar{Y}$ where

$$EY = \int_{\theta}^{\infty} y \cdot e^{-(y-\theta)} \, dy$$
$$= e^{\theta} \cdot \int_{\theta}^{\infty} y \cdot e^{-y} \, dy$$
$$= e^{\theta} \cdot (\theta e^{-\theta} + e^{-\theta})$$
$$= \theta + 1$$

Therefore, $\hat{\theta}_{mom} = \bar{Y} - 1$.

(b) $$L(\theta \mid \underset{\sim}{y}) = \prod_{i=1}^{n} e^{-(y_i - \theta)} I(y_i > \theta)$$
$$= e^{-\Sigma(y_i - \theta)} \cdot I(y_{(1)} > \theta)$$
$$= \underbrace{e^{n\theta} \cdot e^{-\Sigma y_i}}_{\text{increasing function of } \theta} \cdot I(y_{(1)} > \theta)$$

Therefore, MLE of θ is: $\hat{\theta} = Y_{(1)}$, the MLE of $EY = \theta + 1$ is $Y_{(1)} + 1$.

(c) $E(\hat{\theta}_{mom}) = E(\bar{Y}) - 1 = \theta \Rightarrow \hat{\theta}_{mom}$ is unbiased.

$E(Y_{(1)}) = \theta + \frac{1}{n}$ (see Take home #4)

$\Rightarrow Y_{(1)} - \frac{1}{n}$ is unbiased for θ.

(d). $MSE(\tilde{\theta})$

$$= Var(\tilde{\theta}) + [Bias(\tilde{\theta})]^2$$

$$= Var(\bar{Y}) + 0$$

$$= \frac{1}{n}$$ (Hint: first find ~~EY~~ EY^2)

$$MSE(\hat{\theta}) = Var(Y_{(1)}) + [Bias(Y_{(1)})]^2$$

$$= \frac{1}{n^2} + \frac{1}{n^2} = \frac{2}{n^2}$$

(e) Again, $L(\theta | \underset{\sim}{y}) = e^{-\Sigma y_i} e^{n\theta} \cdot I(y_{(1)} > \theta)$

The sufficient statistic is $Y_{(1)}$.

(f) show $Y_{(1)}$ is complete:

If $E(g(Y_{(1)})) = 0$ for all θ, then

$$\int_{\theta}^{+\infty} g(u) \cdot n e^{-n(u-\theta)} du$$

$$= n e^{n\theta} \int_{\theta}^{+\infty} g(u) e^{-nu} du$$

$$= 0$$

Take ~~partial~~ derivative on both sides.

$$\frac{d}{d\theta} E(g(Y_{(1)})) = 0$$

$$\Rightarrow \frac{d}{d\theta}(ne^{n\theta}) \underbrace{\int_\theta^\infty g(u) e^{-nu} du}_{0} + ne^{n\theta} \frac{d}{d\theta} \int_\theta^\infty g(u) e^{-nu} du = 0$$

$$\Rightarrow \quad ne^{n\theta} \cdot (-1)\, g(\theta) e^{-n\theta} = 0$$

$$\Rightarrow ng(\theta) = 0 \quad \text{for all } \theta.$$

Therefore $g(u) = 0$. for all u.

Hence, $Y_{(1)}$ is complete sufficient with mean $\theta + \frac{1}{n}$. Based on Lehman Scheffé, $Y_{(1)} - \frac{1}{n}$ is MVUE.

(g) $$f_{Y_{(n)}}(y) = n \cdot f_Y(y) \cdot [F_Y(y)]^{n-1}$$

$$= n \cdot e^{-(y-\theta)} \cdot [1 - e^{-(y-\theta)}]^{n-1}, \quad y > \theta$$

Question 4:

(a) $M_{\frac{2\Sigma Y_i}{\beta}}(t)$

$$= E\left(e^{t \cdot \frac{2\Sigma Y_i}{\beta}}\right) = E\left(e^{\frac{2t}{\beta} \cdot \Sigma Y_i}\right) = \prod_{i=1}^{n} M_{Y_i}\left(\frac{2t}{\beta}\right)$$

$$= \prod_{i=1}^{n}\left(\frac{1}{1-\beta\frac{2t}{\beta}}\right)^2 = \left(\frac{1}{1-2t}\right)^{2n} \sim \chi^2_{4n}.$$

Therefore, $\frac{2\Sigma Y_i}{\beta}$ is a function of data and β, whose dist does not depend on β, it is a pivotal quantity.

(b) Define two points a and b where

$$a = \chi^2_{4n,0.975} = \chi^2_{40,0.975} = 24.43$$

$$b = \chi^2_{40,0.025} = 59.34.$$

$$P\left(24.43 < \frac{2\Sigma Y_i}{\beta} < 59.34\right) = 0.95$$

$$\Rightarrow P\left(\frac{2\Sigma Y_i}{59.34} < \beta < \frac{2\Sigma Y_i}{24.43}\right) = 0.95$$